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CALCULATION OF THE DIFFUSION OF SMALL PARTICLES IN A NON-UNIFORM ATMOSPHERE

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SUMMARY

The downward motion of small particles in an isothermal turbulence free atmosphere has been studied for the case of variable diffusion coefficient, utilizing two models. In the first of these, the diffusion coefficient has been taken as an exponential function of altitude while in the second, the variation has been assumed to be parabolic. In the exponential model an analytic solution for simultaneous diffusion in the vertical and horizontal directions has been obtained; while for the parabolic model, an analytic solution for vertical motion only has been obtained. The calculations, which have been restricted to the exponential case show the development of a profile of constant shape, all points of which move with the same velocity, independent of the mass of the particles. These results, which are derivable from our solution, are in agreement with the numerical work of Banister and Davis. It is also shown that the qualitative nature of the results is independent of the initial distribution

CALCULATION OF THE DIFFUSION OF SMALL PARTICLES IN A NON-UNIFORM ATMOSPHERE

by Milton M. Klein and Kwang Yu

1. Introduction

In recent years the programs of chemical releases and nuclear explosions have stimulated interest in the general problem of diffusive motion of particles in the atmosphere. General analytic solutions are not available because of the variation of density (and hence diffusion coefficient) in the atmosphere. Further complexity is added to the problem because of the general downward drift due to the gravitational field. If the region of interest is not too large, the diffusion coefficient may be considered constant; and, for the case of negligible drift velocity, standard methods of solution may be applied.[1][2] The effect of a drift velocity has been taken into account for the case of constant diffusion and constant drift velocity by Chandrasekhar who uses a change of variable technique to reduce the problem to a standard one in heat conduction.[3]

An investigation of the descent of small particles through an exponential atmosphere by means of numerical solution of the governing equations has recently been made by Banister and Davis.[4] The study was restricted to motion in the vertical plane. One of the striking results of their work is the change in a short time interval of the initial density distribution into a constant profile as it descends through the atmosphere. An analytic solution, motivated by this result, has been developed by Granzow[5] in the form of an infinite series of

associated Laguerre polynomials whose coefficients are obtainable from integrals over the initial distribution.

Because of the general interest in the problem of atmospheric diffusion and the scarcity of detailed solutions when the non uniformity of the atmosphere is taken into account, we have studied the problem in detail for two models. In one of these we have taken the diffusion coefficient as an exponential function of altitude while in the other one the variation is parabolic. For the sake of simplicity, the atmosphere has been assumed isothermal. The exponential model is chosen in accordance with the approximate exponential variation of density of the atmosphere with altitude and is convenient for mathematical analysis when considering an infinite or semi-infinite region. The parabolic model, while it can be used for a semi-infinite space, is more useful for a finite region where the analysis for the exponential model becomes awkward and inconvenient.

We shall restrict ourselves in the present analysis to the case of molecular diffusion and therefore will not consider in any detail the effect of turbulence. Since the atmosphere is reasonably turbulence free down to altitudes of about 100 km, we may assume our calculations are applicable down to 100 km. Any further extension below this altitude should be examined with extreme care. The effect of a turbulent layer upon the particle distribution may be analyzed by methods similar to the one presented in Reference 4.

2. Analysis and Formulation of Model

We consider a non-uniform atmosphere in which the diffusion coefficient D is a known function of altitude z. A foreign gas deposited in the atmosphere will spread in all directions due to diffusion and will drift downward under the action of the gravitational field. The drift velocity will be controlled by the frequency with which the gas molecules collide with the atmosphere. The basic diffusion equation and the dependence of the drift velocity upon the collision frequency may be obtained from the Boltzmann Equation

$$\frac{\partial f}{\partial t} + \vec{c} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{c}} = \left(\frac{\delta f}{\delta t}\right)_{int} + \left(\frac{\delta f}{\delta t}\right)_{ext} \qquad (1)$$

where f is the distribution function for the foreign gas, $\vec{c}$ is the particle velocity, $\vec{r}$ the position vector and $\vec{F}$ the external force per unit mass. The collision term $\left(\frac{\delta f}{\delta t}\right)_{int.}$ is the effect upon f due to the internal collisions among the gas particles while $\left(\frac{\delta f}{\delta t}\right)_{ext}$ is the effect due to collisions between the gas and ambient air particles Since the gas particle density becomes quite low after a short period of time, we shall neglect the self-collision term $\left(\frac{\delta f}{\delta t}\right)_{int.}$ In our case, the only external force is the gravitational field which acts in the negative z direction so that Equation (1) may be written in the form

$$\frac{\partial f}{\partial t} + \vec{c} \cdot \frac{\partial f}{\partial \vec{r}} - g\frac{\partial f_1}{\partial c_3} = \left(\frac{\delta f}{\delta t}\right)_{ext} \qquad (2)$$

where c_3 is the z component of the velocity

We now obtain our macroscopic equations of motion by multiplying Equation (2) by an appropriate power of the velocity and integrating over the velocity space. For our purposes it will be sufficient to take the zeroth and first powers of the velocity. We then obtain

$$\int \frac{\partial f}{\partial t}\, d\vec{c} + \int \vec{c} \cdot \frac{\partial f}{\partial \vec{r}}\, d\vec{c} - g \int \frac{\partial f}{\partial c_3}\, d\vec{c} = \int \left(\frac{\delta f}{\delta t}\right)_{ext.} d\vec{c} \qquad (3)$$

$$\int \frac{\partial f}{\partial t}\, \vec{c}\, d\vec{c} + \int \left(\vec{c} \cdot \frac{\partial f}{\partial \vec{r}}\right) \vec{c}\, d\vec{c} - g \int \frac{\partial f}{\partial c_3}\, \vec{c}\, d\vec{c} = \int \left(\frac{\delta f}{\delta t}\right)_{ext}\, \vec{c}\, d\vec{c} \qquad (4)$$

In Equation (3), since integration over velocity is independent of time and space, we may write

$$\int \frac{\partial f}{\partial t}\, d\vec{c} = \frac{\partial}{\partial t} \int F\, d\vec{c} = \frac{\partial \eta}{\partial t}$$

$$\int \vec{c} \cdot \frac{\partial f}{\partial \vec{r}}\, d\vec{c} = \nabla \cdot \int \vec{c}\, f\, d\vec{c} = \nabla \cdot (\eta v)$$

$$\int \frac{\partial f}{\partial c_3}\, d\vec{c} = \int_{-\infty}^{\infty} d c_1 \int_{-\infty}^{\infty} d c_2 \int_{-\infty}^{\infty} \frac{\partial f}{\partial c_3}\, d c_3 = \int_{-\infty}^{\infty} d c_1 \int_{-\infty}^{\infty} d c_2 \Big[f \Big]_{c_3=-\infty}^{c_3=\infty} = 0$$

where the particle density η and the macroscopic velocity v are defined by

$$\eta = \int f\, d\vec{c}, \quad \vec{v} = \frac{1}{\eta} \int f\, \vec{c}\, d\vec{c} \,. \qquad (5)$$

Since the velocity distribution vanishes at the limits, the term in $\frac{\partial f}{\partial c_3}$ gives zero contribution. The collision integral in Equation (3) may be shown to vanish when integrated over the velocity space. The macroscopic form of Equation (3) therefore gives the usual equation of continuity

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta v) = 0 \qquad (6)$$

In Equation (4) it is convenient to consider a typical component of $\vec{c}$, say c_α and obtain

$$\int \frac{\partial f}{\partial t} c_\alpha \, d\vec{c} = \frac{\partial}{\partial t}(\eta v_\alpha)$$

$$\int \vec{c} \cdot \frac{\partial f}{\partial \vec{r}} c_\alpha \, d\vec{c} = \int \nabla \cdot (\vec{c} f) c_\alpha \, d\vec{c}$$

$$= \nabla \cdot \int \vec{c} \, c_\alpha f \, d\vec{c}$$

$$\int \frac{\partial f}{\partial c_3} c_\alpha \, d\vec{c} = \int_{-\infty}^{\infty} d c_1 \int_{-\infty}^{\infty} d c_2 \int \frac{\partial f}{\partial c_3} c_\alpha \, d c_3$$

for $\alpha \neq 3$ the integral over $d c_3$ gives

$$\int \frac{\partial f}{\partial c_3} c_\alpha \, d c_3 = \Big[c_\alpha f \Big]_{-\infty}^{\infty} = 0$$

For $\alpha = 3$ we get

$$\int \frac{\partial f}{\partial c_3} c_3 \, d c_3 = \Big[c_3 f \Big]_{-\infty}^{\infty} - \int f \, d c_3$$

and

$$\int \frac{\partial f}{\partial c_3} c_3 \, d\vec{c} = - \int f \, d\vec{c} = -\eta \tag{7}$$

We may, therefore, write Equation (4) in the form

$$\frac{\partial}{\partial t}(\eta v_\alpha) + \nabla \cdot \int \vec{c} \, c_\alpha f \, d\vec{c} + g\eta \, \delta_{\alpha 3} = \int \left(\frac{\delta f}{\delta t}\right)_{ext.} c_\alpha \, d\vec{c} \tag{8}$$

Because of the complicated form of Equation (8), the direct solution of our problem by means of Equations (6) and (8) is a formidable task. We shall, therefore, utilize Equation (8) to give us an approximate expression for the drift velocity and then solve Equation (6). In Equation (8) we

note that $\left(\frac{\delta f}{\delta t}\right)_{ext.}$ is proportioned to f and to the collision frequency which is a function of a velocity and position. The dependence upon the velocity distribution is quite weak[6] and may be ignored here; we may, therefore, write

$$\int \left(\frac{\delta f}{\delta t}\right)_{ext.} c_\alpha \, d\vec{c} \simeq \int \nu(\vec{r}, \vec{v}) \, f \, c_\alpha \, d\vec{c}$$

$$\simeq \nu \int f \, c_\alpha \, d\vec{c}$$

$$= \nu \eta v_\alpha$$

where ν is the average value of the collision frequency over the velocity space. The velocity v_α is the sum of the diffusion velocity $v_{\alpha D}$ and the drift velocity $v_{\alpha g}$. Since, the drift velocity goes to zero for $g = 0$, We may identify the drift velocity term on the right-hand side of Equation (8) with the term containing g on the left side. We therefore have

$$g \eta \delta_{\alpha 3} = \nu \eta \, v_{\alpha g} \tag{9}$$

from which

$$v_{\alpha g} = \frac{g \delta_{\alpha 3}}{\nu} \tag{10}$$

We now express the velocity v in Equation (6) as the sum of the diffusion velocity v_D and the drift velocity v_g and write

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \vec{v}_D + \eta \vec{v}_g) \tag{11}$$

where, for our case, $\vec{v}_g$ has only a z component v_{3g}, given by Equation (10). The diffusion velocity v_D is given by Fick's law[7]

$$\dot{v}_D n_1 = -D \nabla n_1 \qquad (12)$$

where D is the diffusion coefficient. We may, therefore, write Equation (11) in the form

$$\frac{\partial \eta}{\partial t} - \nabla \cdot (D \nabla \eta) + \nabla \cdot (\eta \vec{v}_g) = 0 \quad (13)$$

3. <u>Analysis of the Diffusion Equation</u>

The solution of Equation (13) for the general three-dimensional case is quite difficult. It is possible to separate Equation (13) into two equations; one governing the motion in the horizontal plane and the other governing the motion in the vertical plane, provided we assume that the coupling between the two motions is weak. The significance of this approximation and an interpretation of the resulting solutions will be presented later. A convenient and useful variable whose differential equation describing the vertical motion is not too complicated is provided by the particle density integrated with respect to x and y, i.e., the number of particles per unit z length. To obtain this equation we first express Equation (13) in cartesian form and, noting that v_g is a function of z only, obtain

$$\frac{\partial \eta}{\partial t} - \frac{\partial}{\partial x}\left(D \frac{\partial \eta}{\partial x}\right) - \frac{\partial}{\partial y}\left(D \frac{\partial \eta}{\partial y}\right) - \frac{\partial}{\partial z}\left(D \frac{\partial \eta}{\partial z} - v_g \eta\right) = 0 \qquad (14)$$

Integrating with respect to x and y from $-\infty$ to $+\infty$ and noting that $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$ vanish at the limits, we obtain

$$\frac{\partial n_1}{\partial t} - \frac{\partial}{\partial z}\left(D \frac{\partial n_1}{\partial z} - v_g n_1\right) = 0 \qquad (15)$$

where

$$n_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta \, dx\, dy$$

is the number of particles per unit z length. We now look for a solution where n_1 (z) describes the vertical motion and write

$$\eta(x,y,z) = n_1(z)\, n_2(x,y,z) \tag{16}$$

where, since we are assuming weak coupling, n_2 has only a weak z dependence. Substituting Equation (16) in Equation (13) yields

$$\frac{n_1 \partial n_2}{\partial t} + \frac{n_2 \partial n_1}{\partial t} = n_2 \nabla \cdot (D \nabla n_1 - v_g n_1) + \nabla \cdot (D\, n_1 \nabla n_2)$$

$$+ \nabla n_2 \cdot (D \nabla n_1 - v_g n_1) \tag{17}$$

which, by virtue of Equation (15), can be simplified to

$$\frac{n_1 \partial n_2}{\partial t} = \nabla \cdot (D\, n_1 \nabla n_2) + \nabla n_2 \cdot (D \nabla n_1 - v_g n_1). \tag{18}$$

Since D and v_g are functions of z only, we may write Equation (18) in the scaler form

$$\frac{n_1 \partial n_2}{\partial t} = \frac{\partial}{\partial z}\left(D n_1 \frac{\partial n_2}{\partial z}\right) + \left(D \frac{\partial n_1}{\partial z} - v_g n_1\right) \frac{\partial n_2}{\partial z} + D n_1 \left(\frac{\partial^2 n_2}{\partial x^2} + \frac{\partial^2 n_2}{\partial y^2}\right). \tag{19}$$

In view of the assumption of weak coupling, we may neglect the terms involving the derivatives of n_2 with respect to z to obtain

$$\frac{\partial n_2}{\partial t} = D \left(\frac{\partial^2 n_2}{\partial x^2} + \frac{\partial^2 n_2}{\partial y^2}\right). \tag{20}$$

Since D is a function of z only, Equation (20) represents the diffusion of material in the x-y plane for a constant diffusion coefficient whose value is determined by the particular altitude at which the diffusion is taking place. A solution of Equation (20) for the case of a source

in the x-y plane is given by[8]

$$n_2 = \frac{M}{4\pi Dt} \exp\left[- \frac{(x^2 + y^2)}{4Dt} \right] \qquad (21)$$

where

$$M = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n_2 \, dx dy$$

is the source strength. Because of the definition of n_2 (Equation (16)), the value of M for our case equals unity.

At this point, we consider the significance of the assumption of weak coupling. If the motion is uncoupled, then in a time t a particle will diffuse a given amount in the x-y plane independent of its downward motion. Actually the particles move in a curved path toward lower values of the diffusion coefficient so that the extent of travel in the x-y plane is actually less than that obtained under the assumption of no coupling. Thus, the diffusion front calculated under our assumption of weak coupling will have a larger extent in the x-y direction than the true diffusion front. The extent of diffusion along the z axis will be correct since Equation (15) for this motion is exact.

The three-dimensional diffusion equation may be treated from an alternative point of view. We first write Equation (13) in the form

$$\frac{\partial \eta}{\partial t} - \frac{\partial}{\partial z}\left(D \frac{\partial \eta}{\partial z} - v_g \eta\right) - D\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) = 0 \qquad (22)$$

and then Fourier transform Equation (22) with respect to x and y to obtain

$$\frac{\partial n_f}{\partial t} - \frac{\partial}{\partial z} \left(D \frac{\partial n_f}{\partial z} - v_g n_f\right) - D\,(\xi^2 + \eta^2)\,n_f = 0 \qquad (23)$$

where

$$n_f = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{N}(\vec{r},t) \exp\left[i(x\xi + y\eta)\right] dx\,dy \qquad (24)$$

and we have made use of the boundary condition that $\mathcal{N}$, $\frac{\partial \mathcal{N}}{\partial x}$, and $\frac{\partial \mathcal{N}}{\partial y}$ vanish at infinity. We note that for the special case $\xi = \eta = 0$, the transformed density function n_f reduces to the density along the z axis, $n_1(z)$. Hence, the density $n_1(z)$ is obtainable as a special case of the Fourier transform function n_f. We shall work with Equation (23) rather than Equation (15) and see to what extent we can solve for the general three-dimensional density $\mathcal{N}$ without appealing to separation of variables and a perturbation procedure for obtaining the diffusion in the horizontal plane.

At this point, it is convenient to express our equations in dimensionless form; we therefore set

$$\vec{r}\,' = \vec{r}/L \qquad (25)$$

$$t' = t/T$$

where L and T are characteristic units of length and time and write Equation (22) in the form

$$\frac{\partial \mathcal{N}}{\partial t'} - \frac{T}{L^2} \frac{\partial}{\partial z'} \left(D \frac{\partial \mathcal{N}}{\partial z'} - \frac{L}{T} v_g \mathcal{N}\right) - D \frac{T}{L^2} \left[\frac{\partial^2 \mathcal{N}}{\partial x'^2} + \frac{\partial^2 \mathcal{N}}{\partial y'^2}\right] = 0 \qquad (26)$$

The Fourier transform of Equation (26) with respect to x' and y yields in place of Equation (23)

$$\frac{\partial n_f}{\partial t'} - \frac{T}{L^2}\frac{\partial}{\partial z'}\left(D\frac{\partial n_f}{\partial z'} - L v_g n_f\right) + \frac{DT}{L^2}\left(\xi'^2 + \eta'^2\right) n_f = 0 \qquad (27)$$

where n_f is given by

$$n_f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta(\vec{r}, t') \exp\left[i(x'\xi' + y'\eta')\right] dx'dy' \qquad (28)$$

We shall take the z axis as positive upward so that the drift velocity v_g in Equation (27) is negative. We therefore replace v_g by $-u_g$ where u_g is positive. It is also convenient to express u_g in terms of D by noting that, from elementary kinetic theory

$$D = \frac{1}{3} v \lambda = \frac{1}{3}\frac{v^2}{\nu} \qquad (29)$$

where v is the mean thermal speed and λ the mean free path. Since u_g from Equation (10) $= g/\nu$, it follows that

$$\frac{u_g}{D} = \frac{3g}{v^2} \qquad (30)$$

and is constant for an isothermal atmosphere.

We may then write Equation (27) in the form

$$\frac{\partial n_f}{\partial t'} - \frac{T}{L^2}\frac{\partial}{\partial z'}\left[D\left(\frac{\partial n_f}{\partial z'} + L\frac{3g}{v^2} n_f\right)\right] + \frac{DT}{L^2}\left(\xi'^2 + \eta'^2\right) n_f = 0 \quad (31)$$

4. Exponential Distribution

We consider now the analysis of Equation (31) for the case of an exponential distribution of diffusion coefficient with altitude We take the origin at the earth's surface and unite

$$\frac{D}{D_o} = e^{z/H} \tag{32}$$

where D_o is a reference value at the origin and H is the scale height Since we have assumed an isothermal atmosphere, H will be taken as constant The differential Equation (31) then becomes

$$\frac{\partial n_f}{\partial t'} - \frac{T}{L^2} D_o \frac{\partial}{\partial z'} \left[\exp (z/H)\left(\frac{\partial n_f}{\partial z'} + L \frac{3g}{v^2} n_f\right)\right] - \frac{D_o T}{L^2} \exp (z/H)(\xi'^2 + \eta'^2) = 0 \tag{33}$$

We note that Equation (33) simplifies considerably with regard to the coefficients if we take H as the unit of length L and H^2/D_o as the unit of time T; we then have

$$\frac{\partial n_f}{\partial t'} - \frac{\partial}{\partial z'}\left[\exp (z')\left(\frac{\partial n_f}{\partial z'} + H \frac{3g}{v^2} n_f\right)\right] - \exp (z') (\xi'^2 + \eta'^2) n_f = 0 \tag{34}$$

Noting that $v^2 = \frac{3kT}{m}$ and $H = \frac{kT}{m_a g}$ where m and m_a are the masses of the foreign gas and the atmospheric molecules; and dropping primes for convenience for the remainder of Section 4, we may write Equation (34) as

$$\frac{\partial n_f}{\partial t} - \frac{\partial}{\partial z}\left[\exp (z)\left(\frac{\partial n_f}{\partial z} + a n_f\right)\right] - \exp (z)(\xi^2 + \eta^2) n_f = 0 \tag{35}$$

where $a = \frac{3gH}{v^2} = \frac{gH}{v_o D_o}$ measures the ratio of the masses $\frac{m}{m_a}$.

We now take the Laplace transform of Equation (35) and obtain

$$\frac{d^2N}{dz^2} + (a+1)\frac{DN}{dz} + \left[a - \xi^2 - \eta^2 - p\exp(-z)\right] N = n_f(z,o)\exp(-z) \tag{36}$$

where $N(z,\xi,\eta,p) = \int_0^\infty \exp(-pt)\, n_f(z,\xi,\eta,t)\, dt$

and $n_f(z,0)$ is the initial density distribution. It is convenient to choose for the initial density distribution a delta function. The corresponding solution for an arbitrary initial distribution can easily be obtained by using the solution for the delta function as a Green's function and integrating over the desired distribution. We therefore take initially

$$\eta(\vec{r},o) = \frac{n_o}{H^3}\,\delta(\vec{r} - \vec{r}_o) \tag{37}$$

where n_o is the total number of particles. We then obtain for the transformed initial condition

$$n_f(z,o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{n_o}{H^3}\,\delta(z - z_o)\,\delta(x - x_o)\,\delta(y - y_o)\exp(ix\xi + iy\eta)\,dx\,dy$$

$$n_f(z,o) = \frac{n_o}{H^3}\,\delta(z - z_o)\exp(ix_o\xi + iy_o\eta) \tag{38}$$

Our differential equation then becomes

$$\frac{d^2N}{dz^2} + (a+1)\frac{dN}{dz} + \left[a - \xi^2 - \eta^2 - p\exp(-z)\right] N = -\frac{n_o}{H^3}\exp(ix_o\xi + iy_o\eta - z)\,\delta(z - z_o) \tag{39}$$

which reduces to

$$\frac{d^2M}{dz^2} + (a + 1)\frac{dM}{dz} + \left[a - \xi^2 - \eta^2 - p\exp(-z)\right] M = -\delta(z - z_o) \qquad (40)$$

by use of

$$M = \frac{H^3}{n_o} N \exp\left[- i(x_o\xi + y_o\eta) + z_o\right]$$

We now change the independent variable z by $\zeta = \exp(-z/2)$ to obtain

$$\zeta^2 \frac{d^2M}{d\zeta^2} - (2a - 1)\frac{dM}{d\zeta} + 4(a - \xi^2 - \eta^2 - p\zeta^2)M = -2\zeta_o\,\delta(\zeta - \zeta_o) \qquad (41)$$

and further introduce

$$Q = \zeta^{-(a-1)}M$$

to yield

$$\zeta^2 \frac{d^2Q}{d\zeta^2} + \zeta\frac{dQ}{d\zeta} - (\beta^2 + 4p\,\zeta^2)\,M = -2\zeta_o^{-a}\,\delta(\zeta - \zeta_o) \qquad (42)$$

where

$$\beta^2 = (a - 1)^2 + 4(\xi^2 + \eta^2)$$

Since $\zeta = \exp(-z/2)$, Q is defined for $0 < \zeta \leq 1$. We now define the analytic continuation of Q into the region $\zeta > 1$ by Equation (42). The right-hand side of Equation (42) is zero at $\zeta = 1$ so that we shall require Q, its first and second derivatives are continuous at $\zeta = 1$ and $Q \to 0$ as $\zeta \to \infty$. This corresponds to the assumption that the particle density approaches zero as $z \to -\infty$. Although the physical space covers only positive values of z, the particle density is not significantly affected by the boundary conditions at $z = 0$ until the majority of particles reach the ground level. Furthermore, the presence of turbulence below 100 km renders invalid any molecular diffusion model

below this level and, in addition, the turbulent layer behaves like a particle sink. We, therefore, expect our present solution to be a good approximation to the actual situation down to the 100 km level.

We now apply the Hankel transform to Equation (42) and obtain as the final solution in the transformed space

$$\bar{Q}(s,p) = 2\,\zeta_o^{-a}\,\frac{J_\beta(s\zeta_o)}{s^2 + 4p} \tag{43}$$

where

$$\bar{Q} = \int_o^\infty \zeta\, J_\beta(s\zeta)\, Q(\zeta,p)\, d\zeta \tag{44}$$

and we have assumed $\zeta Q'$ and $\zeta Q \to 0$ as $\zeta \to 0, +\infty$. Although the actual boundary conditions are stated in terms of M, the assumed boundary conditions imposes no restriction whatsoever, since, if a non trivial solution satisfying the Q boundary conditions is found, it automatically satisfies the M boundary conditions. By the uniqueness theorem for solutions of parabolic differential equations then it is guaranteed to be the only solution.

We now consider the inversion of Equation (43) to obtain our solution in the physical space. Performing the inverse Hankel transform, we obtain

$$Q(\zeta,p) = 2\zeta_o^{-(a+1)} \int_o^\infty \frac{s J_\beta(s\zeta_o)\, J_\beta(s\zeta)}{s^2 + 4p}\, ds$$

$$Q(\zeta,p) = -i\zeta_o^{-(a+1)} \Big[J_\beta(2i\sqrt{p}\zeta)\, H_\beta^{(1)}(2i\sqrt{p}\zeta_o)\, \theta(\zeta_o - \zeta)$$
$$+ J_\beta(2i\sqrt{p}\zeta_o)\, H_\beta^{(1)}(2i\sqrt{p}\zeta)\, \theta(\zeta - \zeta_o) \Big] \tag{45}$$

where θ is the Heaviside step function. The particle density function N

in the p-plane is then

$$N(\zeta,p) = \frac{n_o}{H^3} \exp\left[i(x_o\xi + y_o\eta)\right] \zeta_o^{1-a} \zeta^{1+a} \pi i \left[J_\beta(2i\sqrt{p\zeta}) H_\beta^{(1)}(2i\sqrt{p\zeta_o}) \theta(\zeta_o - \zeta)\right.$$

$$\left. + J_\beta(2i\sqrt{p\zeta_o}) H_\beta^{(1)}(2i\sqrt{p\zeta}) \theta(\zeta - \zeta_o)\right] \quad (46)$$

Applying the inverse Laplace transform to $N(\zeta,p)$

$$n_f(\zeta,t) = \frac{1}{2\pi i} \int_{-i\infty+c}^{i\infty+c} N(\zeta,p) \exp(pt)\, dp, \quad c > 0 \quad (47)$$

we obtain

$$n_f(\zeta,t) = \frac{n_o}{H^3 t} \exp\left[i(x_o\xi + y_o\eta)\right] \zeta_o^{1-a} \zeta^{1+a} \exp\left[-\frac{\zeta^2 + \zeta_o^2}{t}\right] I_\beta\left(\frac{2\zeta\zeta_o}{t}\right)$$

$$(48)$$

The particle density $\eta(\vec{r},t)$ is then given by

$$\eta(\vec{r},t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n_f \exp\left[-i(x\xi + y\eta)\right] d\xi d\eta$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\xi \exp\left[-i(x - x_o)\xi\right] \int_{-\infty}^{\infty} d\eta \exp\left[-i(y - y_o)\eta\right] \cdot$$

$$\frac{M_o}{H^3 t} \zeta_o^{1-a} \zeta^{1+a} \exp\left[-\frac{\zeta^2 + \zeta_o^2}{t}\right] I_\beta\left(\frac{2\zeta\zeta_o}{t}\right) \quad (49)$$

Setting $x - x_o = r\cos\theta$, $y - y_o = r\sin\theta$

$$\xi = \rho\cos\theta; \qquad \eta = \rho\sin\theta \quad (50)$$

and using the integral representation of J_o, we have

$$\int_{-\infty}^{\infty} d\xi \exp\left[-i(x - x_o)\xi\right] \int_{-\infty}^{\infty} d\eta \exp\left[-i(y - y_o)\eta\right] I_{\sqrt{4\xi^2 + 4\eta^2 + (a-1)^2}} (\cdot)$$

$$= \int_o^{\infty} \rho\, I_{\sqrt{4\rho^2 + \alpha^2}} (\cdot)\, J_o(\rho r)\, d\rho$$

where $\lambda = \dfrac{2\zeta\zeta_o}{t}$ and $\alpha = a - 1$. The particle density $\eta(\vec{r},t)$ may therefore be written as

$$\eta(\vec{r},t) = \frac{n_o}{4\pi^2 H^3 t} \zeta_o^{1-a} \zeta^{1+a} \exp\left[-\frac{\zeta_o^2 + \zeta^2}{t}\right] \int_0^{\infty} I_{\sqrt{4\rho^2+\alpha^2}}\left(\frac{2\zeta\zeta_o}{t}\right)$$

$$\cdot J_o\left(\rho\sqrt{(x-x_o)^2 + (y-y_o)^2}\right) d\rho \qquad (51)$$

The integral occurring in Equation (51) is a Hankel transform of order zero and can easily be evaluated in any particular case. As previously indicated, the one-dimensional solution $n_1(z,t)$ may be obtained by setting $\xi = \eta = 0$ in the solution for n_f. We then obtain from Equation (48)

$$n_1(z,t) = \frac{n_o}{Ht} \zeta_o^{1-a} \zeta^{1+a} \exp\left(-\frac{\zeta^2 + \zeta_o^2}{t}\right) I_{a-1}\left(\frac{2\zeta\zeta_o}{t}\right) \qquad (52)$$

Some useful information concerning the late time history of the particles may be obtained from the solution for $n_1(z,t)$. For large values of t, the Bessel function in Equation (52) may be expanded in a power series; and, upon retention of the leading order term, we obtain as the asymptotic solution for $n_1(z,t)$

$$n_1(z,t) = \frac{n_o}{H} \frac{1}{\Gamma(a)} \left(\frac{\zeta^2}{t}\right)^a \exp\left(-\frac{\zeta^2 + \zeta_o^2}{t}\right) \qquad (52a)$$

The exponential function is close to unity for large t so that dependence of the density function n_1 upon z and t occurs only through the single variable $\dfrac{\zeta^2}{t}$. The altitude z is related to the time t by

$$\zeta^2 = \exp(-z) = ct \qquad (52b)$$

where c is a constant which depends upon the density n_1 and parameters a, H, and z_o. We shall find Equations (52a) and (52b) useful in interpreting the results of the calculations.

5. Parabolic Model

We examine now the case where the diffusion coefficient is a parabolic function of altitude; it is convenient to write the diffusion coefficient in the form

$$\frac{D}{D_1} = \left(\frac{z}{b}\right)^2 \tag{53}$$

where b is the altitude at which the gas is released and D_1 is the reference value of D at that altitude. Substituting Equation (53) in Equation (31) yields

$$\frac{\partial n_f}{\partial t} - \frac{T}{L^2}\frac{D_1}{b^2}\frac{\partial}{\partial z'}\left(z^2 \frac{\partial n_f}{\partial z'} + \frac{3g}{v^2} L z^2 n_f\right) - \frac{T}{L^2}\frac{D_1}{b^2} z^2\left(\xi'^2 + \eta'^2\right) n_f = 0 \tag{54}$$

Parallel to the analysis for the exponential case, we now choose b for the unit of length L and $\frac{b^2}{D_1}$ as the unit of time T; we may then write Equation (54) in the form

$$\frac{\partial n_f}{\partial t'} - \frac{\partial}{\partial z'}\left(z'^2 \frac{\partial n_f}{\partial z'} + \frac{3g}{v^2} b z'^2 n_f\right) - z'^2\left(\xi'^2 + \eta'^2\right) n_f = 0 \tag{55}$$

If we apply the Laplace transformation to Equation (55), the inhomogeneous term of the resulting ordinary differential equation is a delta function. Because of some difficulty in finding the corresponding Green's function, we have explored the use of the Lagrange variable

$$h_f = \int_0^{z'} n_f dz' \tag{56}$$

for which the inhomogeneous term is a step function. Because of the

presence of the ξ' and η' terms in Equation (55), however, the transformed equation is an integral rather than differential equation. We have therefore restricted ourselves in the present analysis to the density function $n_1(z,t)$ along the z axis. Setting $\xi' = \eta' = 0$ in Equation (54) and integrate with respect to z', we obtain

$$\frac{\partial h}{\partial t'} - \left(z' \frac{\partial^2 h}{\partial z'^2} + \frac{3g}{v^2} b z'^2 \frac{\partial h}{\partial z'} \right) = 0 \qquad (57)$$

where

$$h = \int_o^{z'} n_1(z',t) dz' \qquad (58)$$

and we have imposed the boundary condition that the flux vanishes at $z = 0$.

Applying the Laplace transform to Equation (57) and omitting primes for the remainder of Section 5 yields

$$z^2 \frac{\partial^2 H}{\partial z^2} + \frac{3gb}{v^2} z^2 \frac{\partial H}{\partial z} \quad pH = -h(z,o) \qquad (59)$$

where

$$H = \int_o^{\infty} h(z,t) \exp(-pt) dt \qquad (60)$$

and $h(z,o)$ is the initial distribution in terms of h. An exact solution of Equation (59) leads to hypergeometric functions which introduce some difficulties in obtaining the Laplace inversion. We have, therefore, simplified Equation (59) by approximating z^2 in the $\frac{\partial H}{\partial z}$ (drift velocity) term by αz where α is a suitable average of z over the interval. Since

we are interested in values of the altitude above 100 km, we expect the approximation to be a useful one. We may then write Equation (59) as

$$z^2 \frac{\partial^2 H}{\partial z^2} + \beta z \frac{\partial H}{\partial z} - pH = h(z,o) \qquad (61)$$

where $\beta = \frac{3g}{v^2} b$. Solving Equation (61) for the complementary function H_o we obtain

$$H_o = Az^{m_1} + Bz^{m_2} \qquad (62)$$

where m_1 and m_2 are given by

$$m_1 = c + \sqrt{c^2 + p}$$

$$m_2 = c - \sqrt{c^2 + p} \qquad (63)$$

and $c = \frac{1 - \beta}{2}$. For the initial distribution, we choose a delta function at $z = 1$, $\frac{n_o}{b} \delta(z - 1)$. The corresponding Lagrange function h is then the Heaviside step function $\theta(1)$. A particular solution H_p of Equation (61) for this inhomogeneous term is

$$H_p = \frac{n_o}{p} \theta(1) \qquad (64)$$

and the complete solution $H = H_o + H_p$ is then

$$H = Az^{m_1} + Bz^{m_2}, \quad z < 1$$

$$H = Cz^{m_1} + Dz^{m_2} + \frac{n_o}{p}, \quad z > 1 \qquad (65)$$

where we have used different constants in the two regions because of the step function at $z = 1$.

We consider now the boundary conditions for our problem. We shall

place perfect reflectors at $z = 0$ and at an upper level $z = d$ and therefore require zero flux at these locations. From Equations (61) and (64) the flux F is given by

$$F = \text{const.}(n_p - n_o(1)) \qquad (66)$$

In addition, we shall make use of the continuity of $n_1(z,t)$ and hence of $h(z,t)$ and $H(z,p)$ for $t > 0$ at $z = 1$. Since the real part of the parameter p is positive, the exponent m_2 in Equation (65) is negative. We therefore require the coefficient B in Equation (65) to vanish to avoid an infinite solution at $z = 0$. Application of our boundary and continuity conditions then yields

$$\left.\begin{aligned} H &= \frac{n_o}{p}\,\frac{c + q - d^{2q}(c - q)}{2qd^{2q}}\, z^{c+q}, \quad z < 1 \\ H &= \frac{n_o}{p} + \frac{n_o}{p}\,\frac{c + q}{2qd^{2q}}\, z^{c}(z^{q} - c^{2q}z^{-q}), \quad z > 1 \end{aligned}\right\} \qquad (67)$$

where $q^2 = c^2 + p$

The inverse Laplace transform of H may then be obtained from a standard table of inverse Laplace transforms.[9] Several of the inverse transforms are expressed as Faltung integrals which can be related to error functions by

$$\int_0^t \frac{1}{\tau^{1/2}} \exp(-a\tau) \exp\left(\frac{b}{-\tau}\right) d\tau = -\sqrt{\frac{\pi}{a}} \left[\exp(2\sqrt{ab}) \operatorname{Erfc}\left(\sqrt{\frac{b}{t}} + \sqrt{at}\right) - \exp(-2\sqrt{ab}) \operatorname{Erfc}\left(\sqrt{\frac{b}{t}} - \sqrt{at}\right)\right] \qquad (68)$$

Similar relations involving higher powers of $\frac{1}{\tau^{\frac{1}{2}}}$ may be obtained from Equation (68) by differentiating with respect to b. The particle density $n_1(z,t)$ is then obtained from $h(z,t)$ by

$$n_1(z,t) = \frac{\partial h}{\partial z}$$

The results obtained may be expressed as follows:

for $z < 1$,

$$\frac{n_1}{n_o} = \frac{z^{c-1}}{4}\left(z\,\frac{\partial I}{\partial z} + cI\right) \qquad (69)$$

$$I = I_1 + I_2 - I_3 + I_4$$

$$I_2 = \left(\frac{d^2}{z}\right)^{-c} \text{Erfc}\,(a_o - a_1) - \left(\frac{d^2}{z}\right)^{c} \text{Erfc}\,(a_o + a_1)$$

$$I_2 = \left(\frac{d^2}{z}\right)^{c} \text{Erfc}\,(a_o + a_1) + \left(\frac{d^2}{z}\right)^{-c} \text{Erfc}\,(a_o - a_1)$$

$$I_3 = \frac{1}{z^{-c}}\,\text{Erfc}\,(a_2 - a_1) - \frac{1}{z^{c}}\,\text{Erfc}\,(a_2 + a_1)$$

$$I_4 = \frac{1}{z^{c}}\,\text{Erfc}\,(a_2 + a_1) + \frac{1}{z^{-c}}\,\text{Erfc}\,(a_2 - a_1)$$

$$z\,\frac{\partial I_1}{\partial z} = c\left(\frac{d^2}{z}\right)^{c} \text{Erfc}\,(a_o + a_1) + c\left(\frac{d^2}{z}\right)^{-c} \text{Erfc}\,(a_o - a_1)$$

$$z\,\frac{\partial I_2}{\partial z} = \left(\frac{d^2}{z}\right)^{c} \frac{1}{\sqrt{\pi t}}\,\exp\left[-(a_o + a_1)^2\right] + \left(\frac{d^2}{z}\right)^{-c} \frac{1}{\sqrt{\pi t}}\,\exp\left[-(a_o - a_1)^2\right]$$

$$-c\,\text{Erfc}\,(a_o + a_1) + c\,\text{Erfc}\,(a_o - a_1)$$

$$z\,\frac{\partial I_3}{\partial z} = \frac{1}{z^{c}}\,c\,\text{Erfc}\,(a_2 + a_1) + \frac{1}{z^{-c}}\,\text{Erfc}\,(a_2 - a_1)$$

$$z \frac{\partial I_4}{\partial z} = -\frac{1}{z^c \sqrt{\pi t}} \exp\left[-(z_2 + z_1)^2\right] + \frac{1}{z^{-c} \sqrt{\pi t}} \exp\left[-(a_2 - a_1)^2\right]$$

$$- \frac{c}{z^c} \operatorname{Erfc}(a_2 + a_1) + \frac{c}{z^{-c}} \operatorname{Erfc}(a_2 - a_1)$$

$$a_o^2 = \frac{1}{4t} \ln \frac{d^2}{z}$$

$$a_1^2 = ct$$

$$a_2^2 = \frac{1}{4t} \ln \frac{1}{z}$$

for $z > 1$,

$$\frac{n_1}{n_o} = \frac{z^{c-1}}{4} \left(z \frac{\partial J}{\partial z} + cJ \right) \qquad (70)$$

where

$$J = J_1 + J_2 - J_3 - J_4 \quad ; \quad J_1 = I_1, \; J_2 = I_2 ,$$

$$J_3 = z^{-c} \operatorname{Erfc}(a_2' - a_1) - z^c \operatorname{Erfc}(a_2' + a_1)$$

$$J_4 = z^c \operatorname{Erfc}(a_2' + a_1) + z^{-c} \operatorname{Erfc}(a_2' - a_1)$$

$$z \frac{\partial J_1}{\partial z} = z \frac{\partial I_1}{\partial z}$$

$$z \frac{\partial J_2}{\partial z} = z \frac{\partial I_2}{\partial z}$$

$$z \frac{\partial J_3}{\partial z} = -c \frac{1}{z^{-c}} \operatorname{Erfc}(a_2' + a_1) - c \frac{1}{z^c} \operatorname{Erfc}(a_2' - a_1)$$

$$z \frac{\partial J_4}{\partial z} = \frac{-c}{z^{-c}} \frac{1}{\sqrt{\pi t}} \exp\left[-(a_2' + a_1)^2\right] - \frac{c}{z^c} \frac{1}{\sqrt{\pi t}} \exp\left[-(a_2' - a_1)^2\right]$$

$$+ \frac{c}{z^{-c}} \mathrm{Erfc}\,(a_2' + a_1) - \frac{c}{z^{-c}} \mathrm{Erfc}\,(a_2' - a_1)$$

$$a_2'^2 = \frac{1}{4t} \ln z$$

Although the solution for the parabolic case has a complicated analytic form, the asymptotic solution for large t is sufficiently simple to draw some interesting and useful conclusions. The form of the solution depends critically upon the value of c; for c positive the leading order term depends upon z but not t; while for c negative, the time-independent leading order term cancels out and the solution becomes time dependent. The results obtained for these two cases may be expressed in the form

$$\frac{n_1}{n_o} = \frac{2c}{d^{2c}} z^{2c-1}, \quad c > 0 \tag{71}$$

$$\frac{n_1}{n_o} = \text{const.} \frac{1}{zt^{\frac{1}{2}}} \exp(ct), \quad c \leq 0$$

For $c > 0$ the solution is in the form of a power law distribution of the particle density with respect to altitude. This result is due to the approximation in the drift velocity term; if the approximation had not been made, we would have obtained an exponential behaviour similar to that obtained in Reference 3. This is not a serious error, however, since we are not interested in the region z close to zero. We may see the effect of our approximation more closely by directly solving the

differential equation for large t. Noting that the time derivative becomes negligible for large t, we may write Equation (54) in the form

$$\frac{\partial}{\partial z}\left(z^2 \frac{\partial n_1}{\partial z} + \frac{3gb}{v^2} z^2 n_1\right) = 0 \tag{72}$$

which, upon noting that the flux vanishes for large t, may be integrated to

$$z^2 \frac{\partial n_1}{\partial z} + \frac{3gb}{v^2} z^2 n_1 = 0 \tag{73}$$

If we approximate z^2 in the drift velocity term by αz, we obtain

$$z^2 \frac{dn_1}{dz} = -\beta z n_1$$

$$n_1 = Az^{-\beta} \tag{74}$$

Using $n_o = \int_o^d n_1 dz$ to evaluate the constant A, we obtain

$$A = \frac{1-\beta}{d^{1-\beta}} n_o$$

so that

$$\frac{n_1}{n_o} = \frac{1-\beta}{d^{1-\beta}} z^{-\beta} = \frac{2c}{d^{2c}} z^{2c-1} \tag{75}$$

in agreement with Equation (71). If we do not make any approximation in the drift velocity term, however, we obtain

$$z^2 \frac{dn_1}{dz} = -\frac{3gb}{v^2} z^2 n_1$$

$$n_1 = \text{const. } \exp\left[-\frac{3gbz}{v^2}\right] \tag{76}$$

which is an exponential distribution similar to that obtained in Reference 3 for the case of constant D and constant u_g.

The case of no gravitational field ($\beta = 0$, $c = 1/2$) yields a particularly simple result. Setting $g = 0$ in Equation (73) or directly from Equation (75), we obtain, as anticipated, the uniform distribution

$$\frac{n_1}{n_o} = \frac{1}{d} \tag{77}$$

The case $c \leq 0$ is not directly obtainable from Equation (73) since the dropping of the time derivative is not valid here in the neighborhood of the origin. This behaviour may be seen clearly from Equation (71) where we note that $n_i \to 0$ for large t except near $z = 0$. We thus have a piling up of particles at $z = 0$; and since the total number is constant, a delta function distribution is obtained for $t = \infty$. This result is not of great significance since it is due to our approximation for the drift velocity term. It does show, however, that in a situation where the gravitational field becomes significantly larger than the diffusion effect, such a piling up of material may be expected

6. Results

Because of the greater simplicity of results and because of more immediate applicability to the physical atmosphere, we have confined our numerical calculations to the exponential model In addition we have restricted ourselves in the present investigation to the one dimensional density function $n_1(z,t)$. Numerical calculations for the three dimensional density function $(\vec{r},t)$ will be presented in a future investigation. The density function $n_1(z,t)$ is presented as a function of altitude z/H for several values of the time $\frac{D_o}{H^2}$ t and the mass ratio a in Figures 1 through 4. The scale height H was taken as 20 km and altitude of release $\frac{z_o}{H}$ as 20 The figures show that the density profile becomes increasingly narrow as a increases, and the maximum value of n_1 increases with a. These results may be anticipated in view of the smaller momentum changes and consequent lesser dispersion of the heavier particles The time taken for a given profile to descend to a given altitude is easily obtained from the figures. As an example, for a = 1 (Figure 1), it takes about 10^{-2} units of $\frac{D_o}{H^2}$ t for an injection at 400 km to descend to the 100 km level. For a representative value of D_o = 400 cm^2/sec, the corresponding physical time is 10^8 sec or about three years a reasonable result.

The most striking feature of the density profile is its shifting by a constant amount as the time parameter changes, the profile remaining unchanged in form. It thus appears that, after an initial time period required to establish the profile, all particles move with the same velocity. These results have been obtained previously in the numerical calculations of Banister and Davis

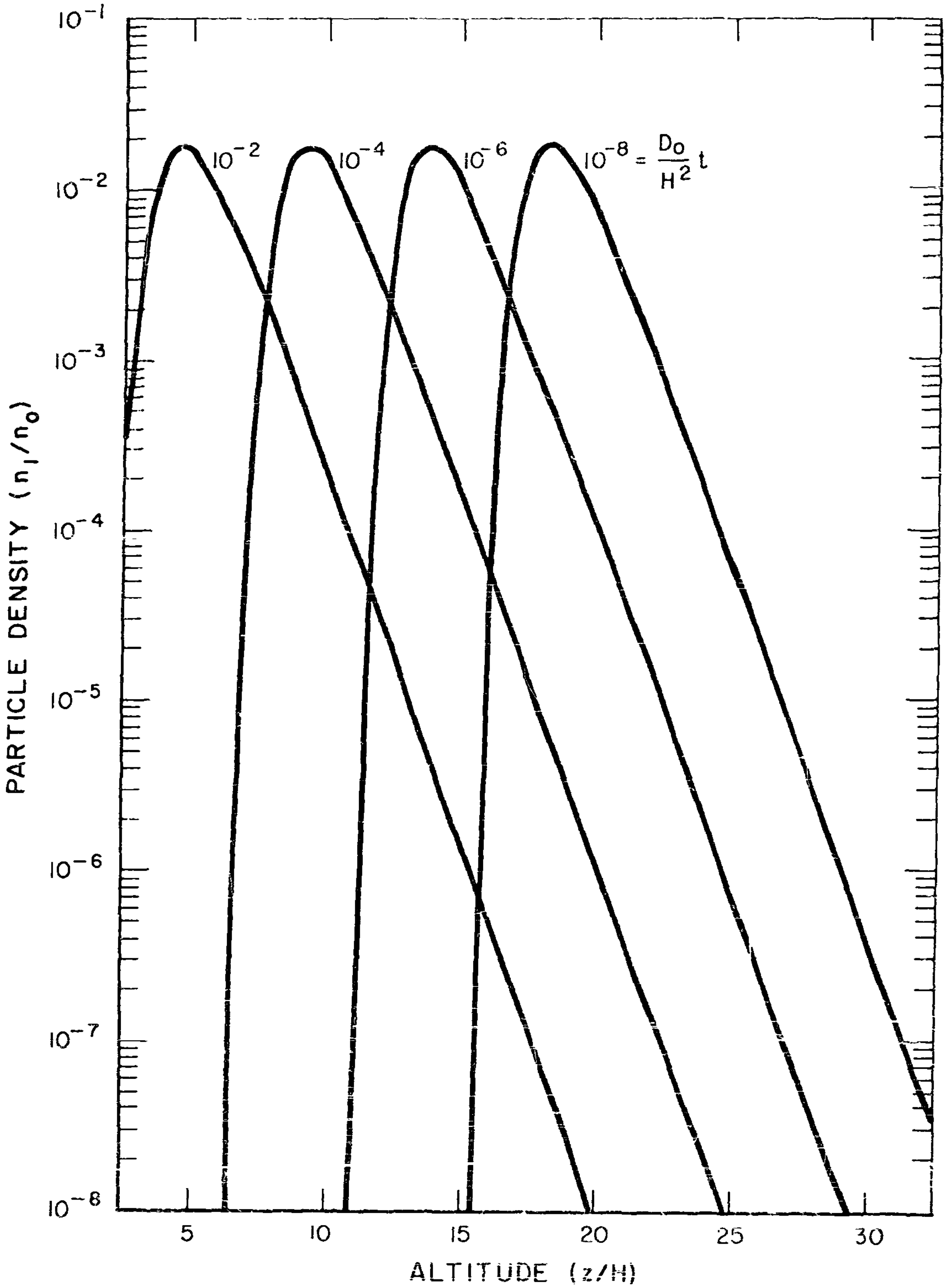

Figure 1. Particle density function $\frac{n_1}{n_o}$ as a function of altitude for several values of time $\frac{D_o}{H^2}$ t; mass ratio a = 1.

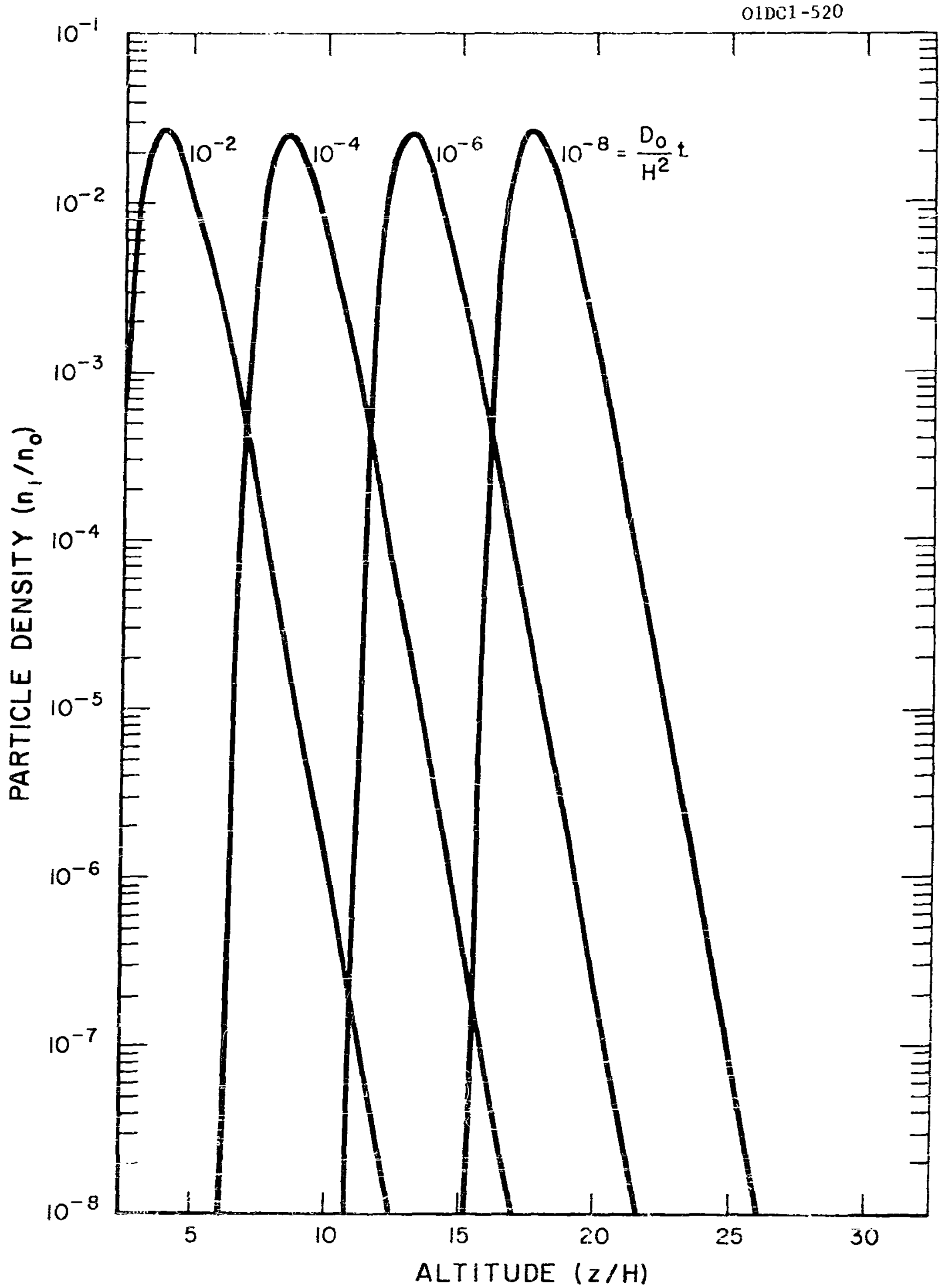

Figure 2. Particle density function $\frac{n_1}{n_o}$ as a function of altitude for several values of time $\frac{D_o}{H^2}$ t; mass ratio a = 2.

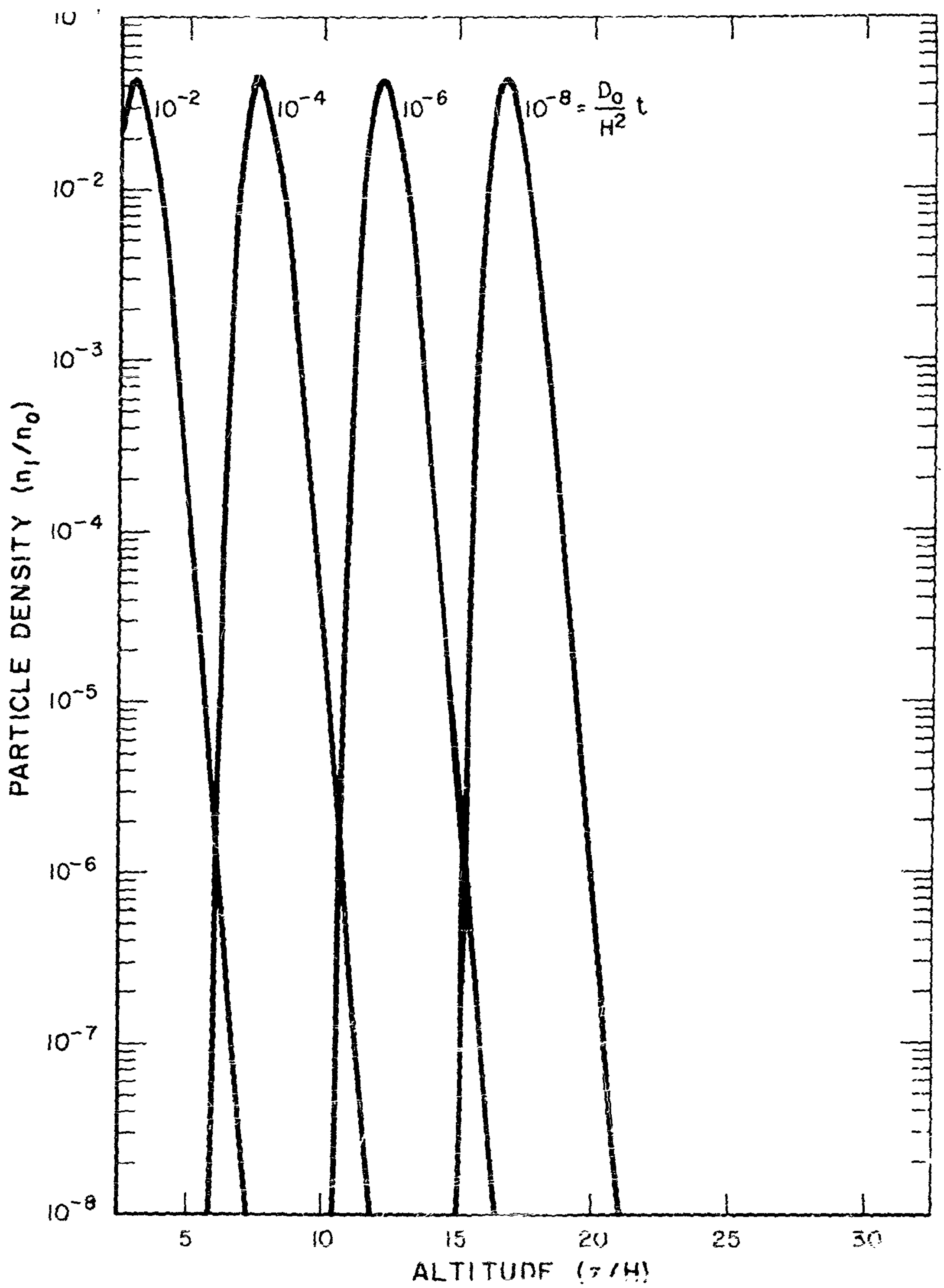

Figure 3. Particle density function $\frac{n_1}{n_0}$ as a function of altitude for several values of time $\frac{D_0}{H^2}$ t; mass ratio a = 5.

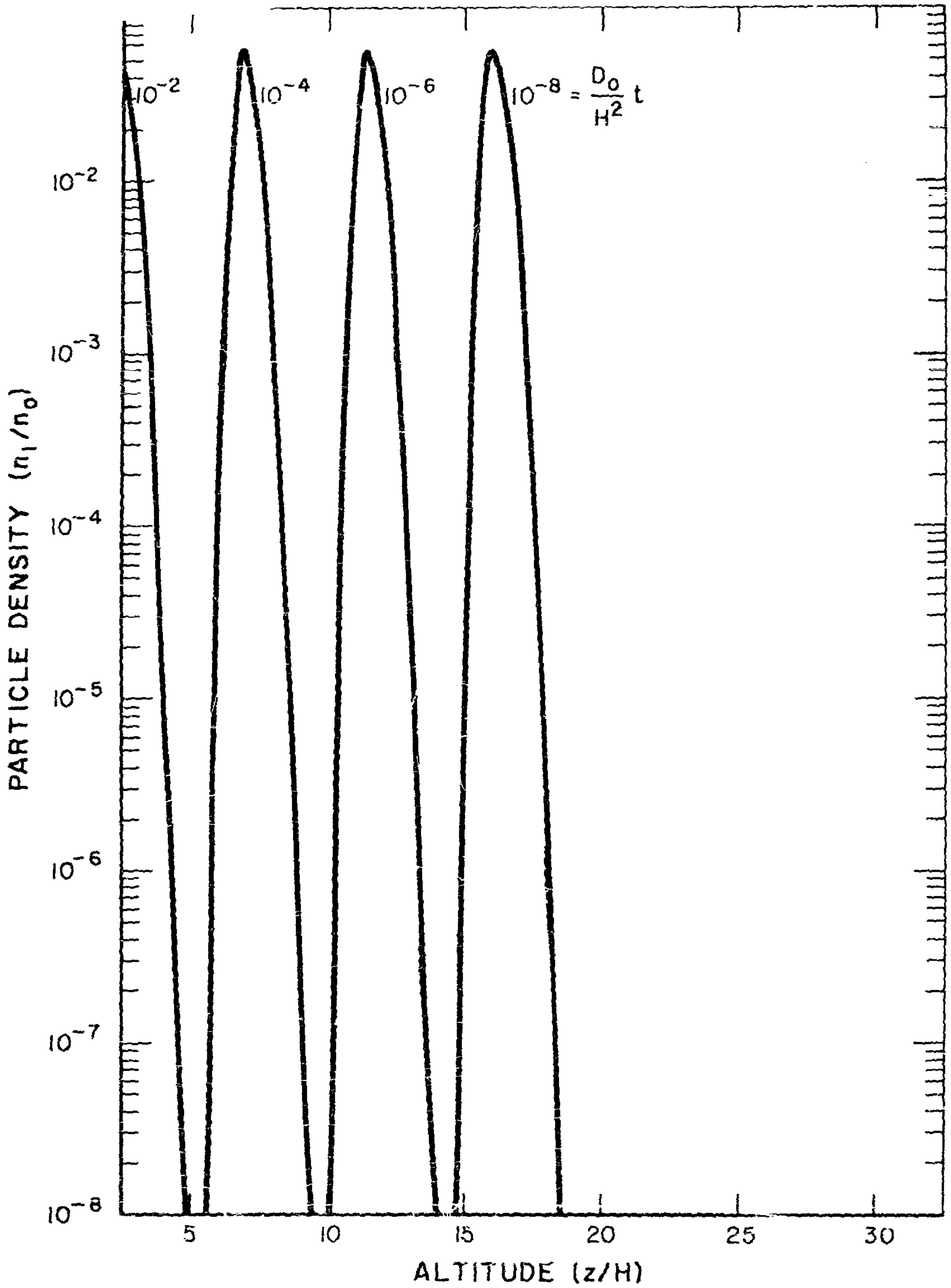

Figure 4. Particle density function $\frac{n_1}{n_o}$ as a function of altitude for several values of time $\frac{D_o}{H^2}t$; mass ratio $a = 10$.

The development of a constant profile and its consequences are easily obtainable from the relation between ζ and t given by Equation (52b). For a fixed value of c, we may write this relation in the form

$$z_1 - z_2 = \ln \frac{t_2}{t_1} \qquad (78)$$

where z_2 and z_1 are typical altitudes and t_2 and t_1 the corresponding times. For our calculations, the ratio of two consecutive times $\frac{t_2}{t_1}$ has been chosen as 100 so that the corresponding difference $z_1 - z_2 = \ln 100 = 4.6$. A check of Figures 1 through 4 shows that the altitude difference between corresponding points on adjacent profiles is always about 4.6 units. The time required to establish a constant profile is easily obtained from Equation (52). Since the argument $\frac{2\zeta\zeta_o}{t}$ of this Bessel function must be small enough to yield the constant profile solution given by Equation (52a), the required time must be somewhat larger than $\zeta\zeta_o$.

The velocity of a particle on a typical profile is, from Equation (52b)

$$\frac{dz}{dt} = - \frac{1}{t} = - \frac{c}{\exp(-z)} \qquad (79)$$

so that the velocities of two particles characterized by the same time, i.e. belonging to the same profile, are equal. We also note that, since the expression for the velocity can be written in terms of t only, particles characterized by different masses will have the same limiting velocities although the corresponding altitudes will be different. This result may easily be seen by noting that all points on a profile have the same velocity. This velocity is, therefore, obtainable from the velocity

at maximum concentration which is entirely a drift velocity and, therefore (see Equation (10)) is independent of mass.

If we write Equation (52b) in the form

$$z = - \ell n c \cdot \ell n t \qquad (80)$$

we see that a plot of z against $\ell n t$ for a fixed value of c should result in a straight line. Since the constant c enters as an additive term, lines characterized by different values of a will be parallel to each other. The constant c takes on a particularly simple form if the maximum value of n_1 is used in Equation (52a). For this case, $\frac{z^2}{t} = a$ so that here $c = a$ and the lines in the z vs. $\ell n t$ plot are now characterized by different values of a. This anticipated result is verified in Figure 5 where we have plotted on semi-log paper the time-altitude history for several values of a of the maximum concentration points of the profiles in Figures 1 through 4.

The initial distribution chosen by Banister and Davis for their investigation was a step function; for our case, a delta function was used. It is easy to show that the foregoing results are independent of the initial distribution. For a delta function, initial distribution $\delta(z - z_o)$ the corresponding solution is the Green's function $G(z,z_o t)$. Using the properties of the Green's function, we may write the solution $n_1(z,t)$ corresponding to any arbitrary initial distribution $f(z)$ in the form

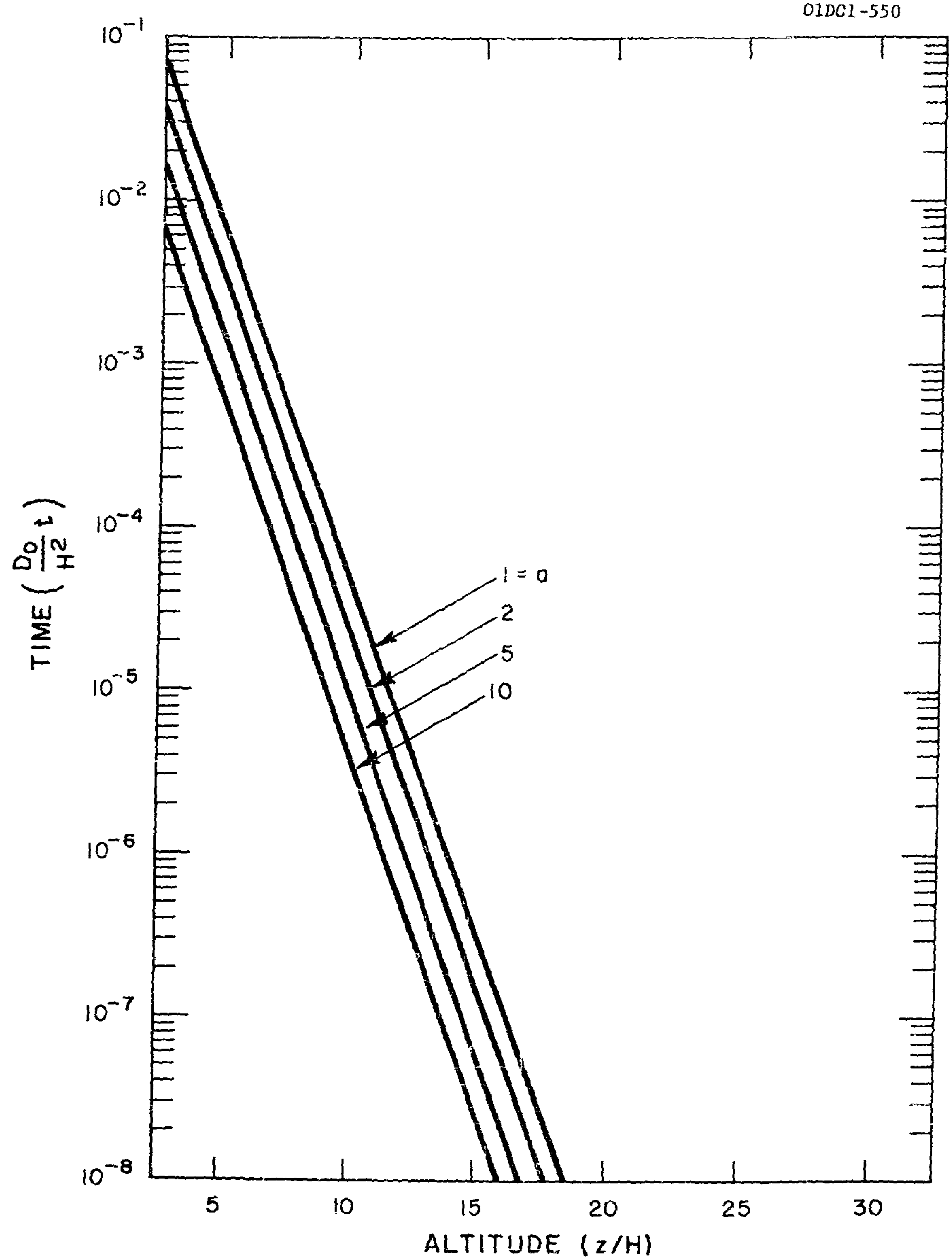

Figure 5. Time-altitude history of maximum concentration points for several values of mass ratio a.

$$\frac{n_1(z,t)}{n_o} = \int_{-\infty}^{\infty} f(z_o)\ G(z,z_o,t)\ dz_o \qquad (81)$$

Using Equation (52a) we note that for large t, G splits into a product of a function F of z and t and a function φ of z_o,

$$G(z,z_o,t) \rightarrow F(z,t)\ \varphi(z_o) \qquad (82)$$

We then have for large t,

$$\frac{n_1(z,t)}{n_o} = \int_{-\infty}^{\infty} F(z,t)\ \varphi(z_o)\ f(z_o)\ dz_o \qquad (83)$$

$$\frac{n_1(z,t)}{n_o} = A\ F(z,t) \qquad (83a)$$

where

$$A = \int_{-\infty}^{\infty} \varphi(z_o)\ f(z_o)\ dz_o \qquad (83b)$$

depends only on the parameters H and a. Thus, the solution retains the same analytic structure in the region of interest, independent of the initial distribution.

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